An Introduction to Functional Data Analysis

Cody Carroll

University of California, Davis

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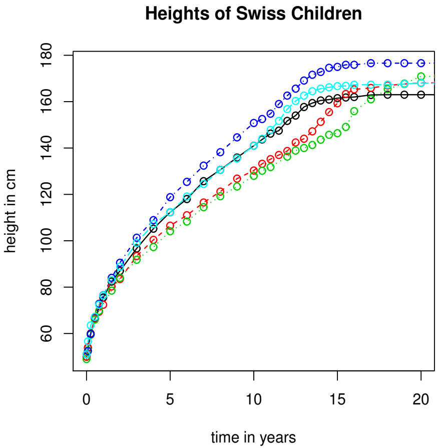
What is functional data?

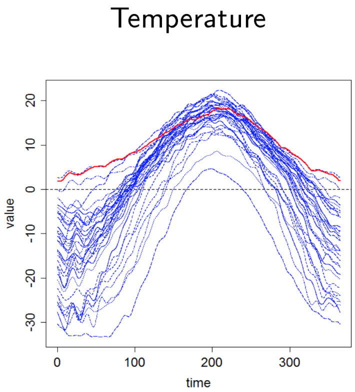
What are the striking features of the following sets of data?



Sample of human growth curves recorded in the Zürich Longitudinal Study.

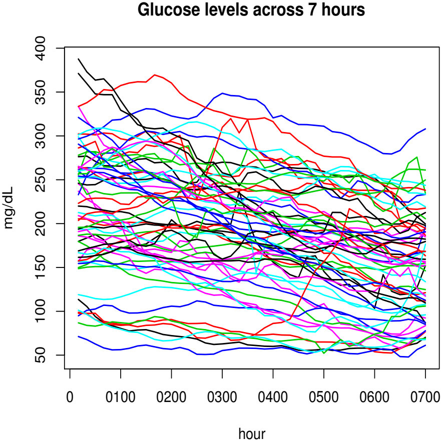
Examples

Temperatures (in $^\circ$C) over the course of a year measured at thirty-five weather stations across Canada.



Examples

An individual's blood glucose levels measured over 7-hour window for 59 days.



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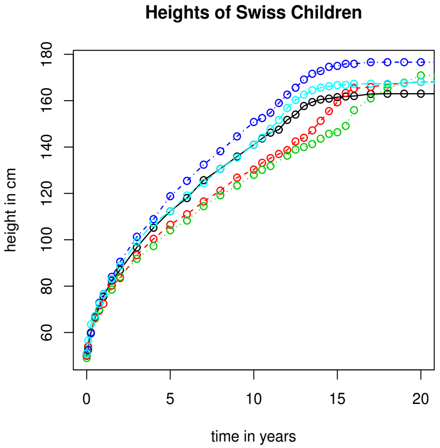


Features:

- similar shape
- repeated observations
- frequency
- smoothness

What is functional data?

- Functional Data: A sample of random functions, with one function per subject.
  - These functions can be curves (1D), images (2D or 3D), or have an higher number of components.
- Characteristics of functional data:
  (i) The "unit" of functional data is a function, $X_i(t)$.
  (ii) Functions are (in theory) $\infty-$dimensional data. There are an infinite number of time points to observe the function on.

Functional Data Basics

In theory, data $X_1, \ldots, X_n$ are i.i.d. copies of a random function $X(t)$.
In practice, the observed data for subject i is actually:

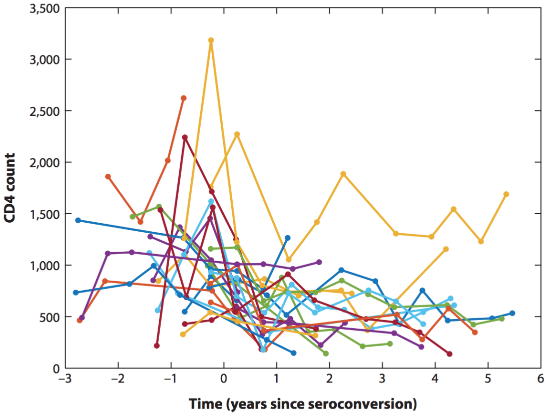
$$Y_{ij} = X_i(t_{ij}) + e_{ij}, \quad j = 1, \ldots, n_i,$$

where $X_i(t)$ is a smooth random function and e_{ij} are independent realizations of random noise for all i, j.

- Regular functional data - All subjects are measured at the same time grid, $t_1, \ldots, t_m \implies$ (often high-dim.) multivariate data.
- Irregular functional data - The measurement schedule for subject i is $t_{i1}, \ldots, t_{in_i} \implies$ longitudinal data.

Longitudinal Example

Irregularly recorded CD4 T lymphocyte count data for 25 HIV+ patients.

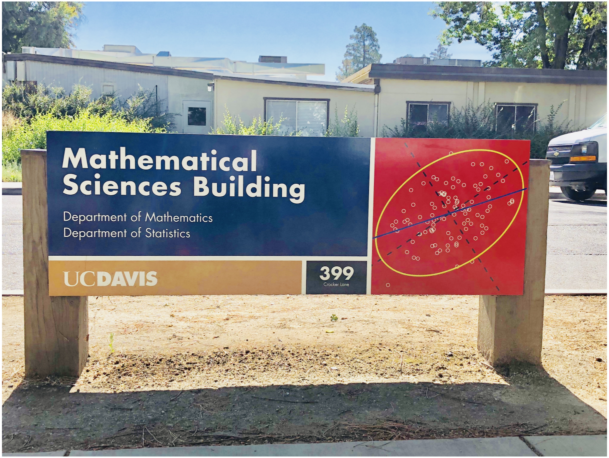


What are we interested in?

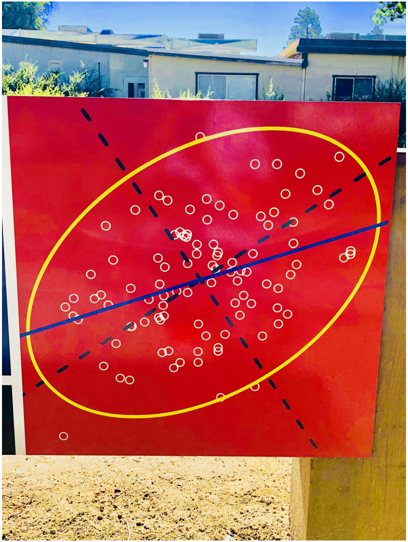
- Estimating the sample functions from finitely many observed points
- Identifying important sources of patterns/variation among the data
- Other more complicated questions

We need a statistical tool to accomplish these tasks. Let's take some inspiration from outside...

Outside this building



Principal Components Analysis (PCA)

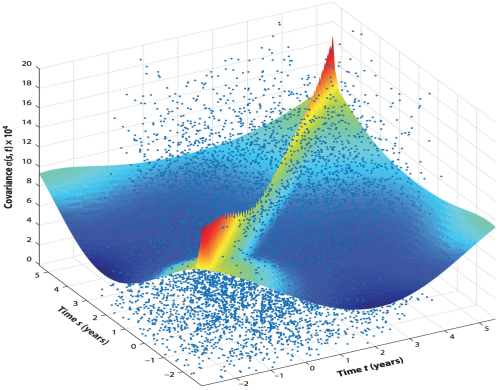


PCA

- PCA takes multivariate data (in $\mathbb{R}^p$) and finds the orthogonal "directions" (vectors) which best explain the variation in the data
- Each observation gets a "score" per direction, indicating how far in that direction the observation lies
- Often we only need 2 or 3 "directions" and their scores for an observation to recreate the data accurately, even when p is large
- Optional: mathematically, these directions are the eigenvectors of the sample covariance matrix of the data

Extend PCA to FPCA

- We want to find the "directions" which explain variation for functional data now
- In functional data, we have covariance surfaces instead of covariance matrices

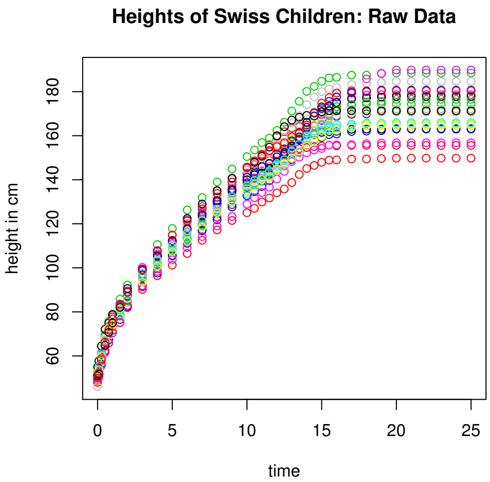


FPCA

- With some math we can obtain the best “directions” from the sample covariance surface which are called eigenfunctions
- Each observation will receive one score per eigenfunction, like before
- Often we only need 2 or 3 eigenfunctions to explain most of the variation, like in PCA
- Also, these eigenfunctions often have physical interpretations

FPCA Example

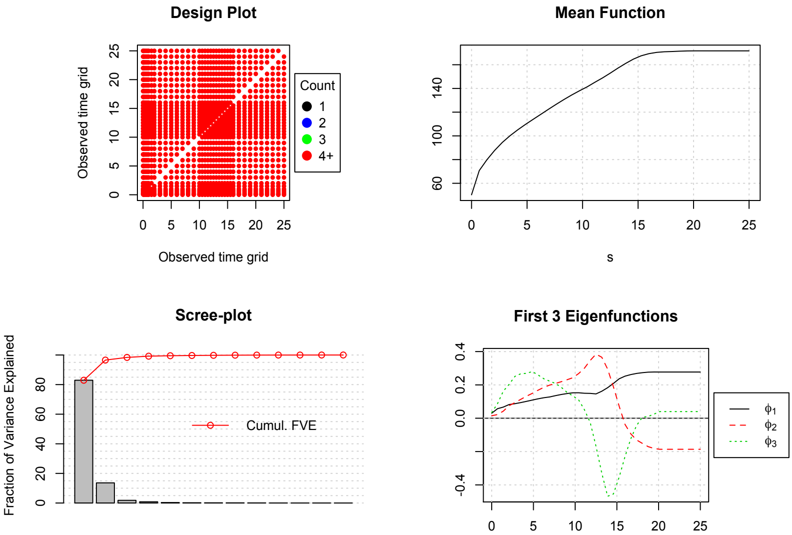
25 human growth curves recorded in the Zürich Longitudinal Study.



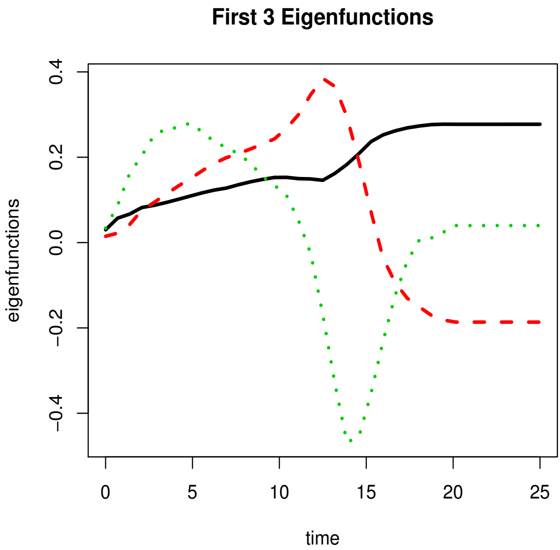
some math later...

FPCA Example

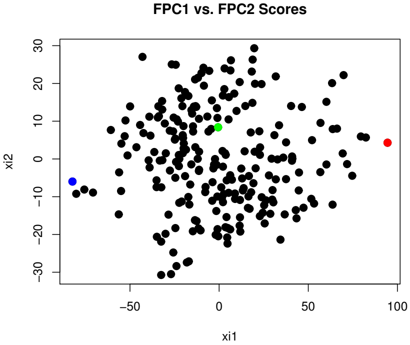
Using the `FPCA()` function in the `fdapace` package:

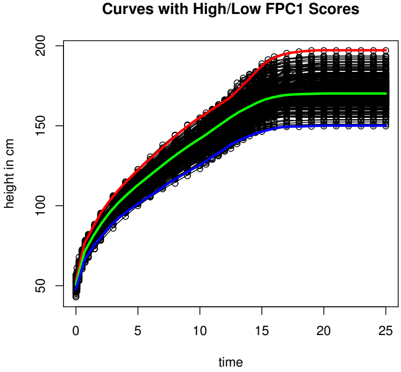


FPCA Example

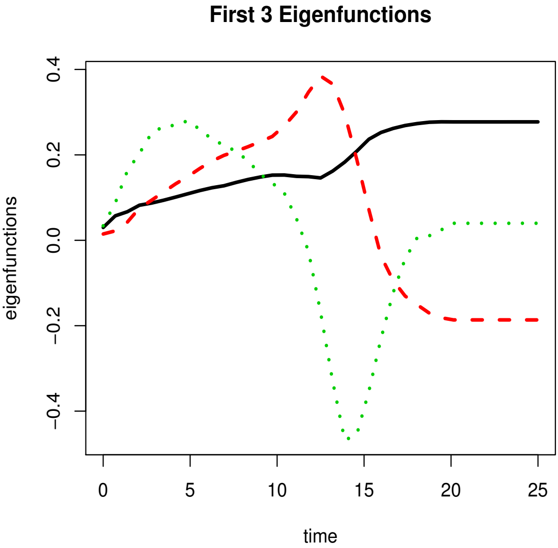


FPCA Example





FPCA Example



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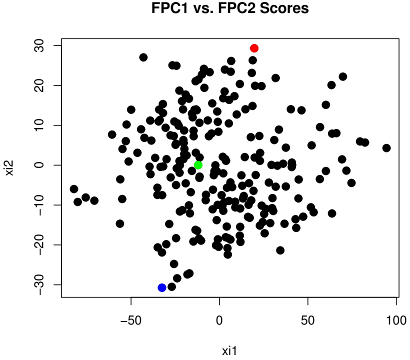

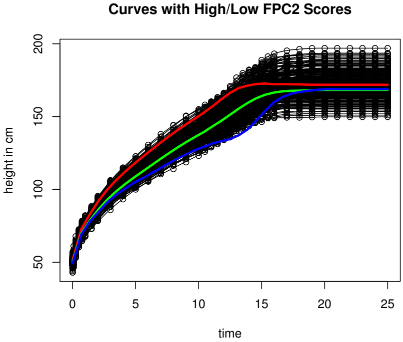

More complicated questions in FDA

- Functional regression:
  - function-on-function
  - function-on-scalar
  - scalar-on-function
- Questions of inference
- Functional registration
- Density estimation

References I

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